Int. J. Heat Mass Transfer. Vol. 19, pp. 1149–1152. Pergamon Press 1976. Printed in Great Britain

APPLICATION OF FORMULA MANIPULATION TO NONEQUILIBRIUM FLUID FLOW

M. D. MIKHAILOV
Applied Mathematics Centre, Sofia C, P.O.B. 384, Bulgaria

(Received 1 April 1976)

Abstract—The problems of nonisothermal fluid flow between parallel plates and in a circular tube are solved for the case when the thermal conductivity and viscosity are functions of temperature. The solutions are obtained using formula manipulation techniques.

NOMENCLATURE

k, thermal conductivity;
m, 0 or 1 for a parallel-plate channel and a circular tube, respectively;
M, $= \left(\frac{\mathrm{d}p/\mathrm{d}z}{m+1}\right)^2 \frac{R^4}{k_0 \mu_0 T_w}$, flow parameter;
$\mathrm{d}p/\mathrm{d}z$, axial pressure gradient;
r, radial coordinate;
R, half-width of the channel or radius of the pipe;
$T(r)$, temperature;
$u(r)$, velocity;
p, s, constants.

Greek symbols
λ, $= T_m/T_w$, dimensionless temperature ratio;
μ, dynamic viscosity;
ρ, $= r/R$, dimensionless space coordinate.

Subscripts
a, average;
w, wall property;
0, reference state;
m, maximum.

Superscripts
bar (dimensionless notation for temperature, velocity, thermal conductivity and dynamic viscosity).

1. INTRODUCTION

RECENT development of programming science permits one to use computers for formula manipulation. Bibliography and detailed information about symbolic and algebraic manipulation technique can be found in [1–4].

The purpose of the present paper is to demonstrate applications of the symbolic manipulation system FORMAL* to heat-transfer problems. For example, the problems of slow viscous incompressible flow between parallel plates and in circular tubes, where the phenomenological coefficients of dynamic viscosity and thermal conductivity are functions of temperature, are solved. In this journal the same problems were treated previously by Butler and Rackley [5].

*FORMAL was originally developed at the Applied Mathematics Centre, Bulgaria.

2. BASIC EQUATIONS AND SOLUTIONS

Consider Poiseuille flow between parallel plates ($m = 0$) and in a circular tube ($m = 1$) at a constant wall temperature T_w, thus any temperature gradient observed is due to the effects of viscous dissipation within the channel. The flow is one-dimensional and the axial pressure gradient $\mathrm{d}p/\mathrm{d}z$ is constant. The momentum equation and boundary conditions can be expressed as

$$\frac{1}{r^m}\frac{\mathrm{d}}{\mathrm{d}r}\left\{\mu(T) r^m \frac{\mathrm{d}u(r)}{\mathrm{d}r}\right\} = \mathrm{d}p/\mathrm{d}z \tag{1}$$

$$\frac{\mathrm{d}u(0)}{\mathrm{d}r} = 0 \tag{2}$$

$$u(R) = 0. \tag{3}$$

Integration of equation (1) from 0 to r and use of condition (2) leads to

$$\frac{\mathrm{d}u(r)}{\mathrm{d}r} = \frac{\mathrm{d}p/\mathrm{d}z}{m+1}\frac{r}{\mu(T)}. \tag{4}$$

The velocity is obtained through integration of equation (4) from r to R and using condition (3)

$$u(r) = -\frac{\mathrm{d}p/\mathrm{d}z}{m+1}\int_r^R \frac{r}{\mu(T)}\mathrm{d}r. \tag{5}$$

The average velocity of the flow is given by

$$u_a = \frac{m+1}{R^{m+1}}\int_0^R r^m u(r)\,\mathrm{d}r. \tag{6}$$

Substituting the velocity $u(r)$ from the solution (5) into equation (6) gives

$$u_a = -\frac{\mathrm{d}p/\mathrm{d}z}{(m+1)R^{m+1}}\int_0^R \frac{r^{m+2}}{\mu(T)}\mathrm{d}r. \tag{7}$$

Combining the above two relation yields

$$\frac{u(r)}{u_a} = R^{m+1}\int_r^R \frac{r}{\mu(T)}\mathrm{d}r \Bigg/ \int_0^R \frac{r^{m+2}}{\mu(T)}\mathrm{d}r. \tag{8}$$

For convenience it is logical to define $\rho = r/R$, $\bar{u} = u(r)/u_a$, $\bar{\mu} = \mu(T)/\mu_0$. Then the nondimensional velocity corresponding to equation (8) is

$$\bar{u}(\rho) = \int_\rho^1 \frac{\rho}{\bar{\mu}(\bar{T})}\,d\rho \Big/ \int_0^1 \frac{\rho^{m+2}}{\bar{\mu}(\bar{T})}\,d\rho. \tag{9}$$

For constant viscosity the expression for the dimensionless velocity takes the form

$$\bar{u}(\rho) = \frac{m+3}{2}(1-\rho^2). \tag{10}$$

The one-dimensional energy equation and boundary conditions may be written as

$$\frac{1}{r^m}\frac{d}{dr}\left\{k(T)r^m\frac{dT(r)}{dr}\right\} + \mu(T)\left(\frac{du(r)}{dr}\right)^2 = 0 \tag{11}$$

$$\frac{dT(0)}{dr} = 0 \tag{12}$$

$$T(R) = T_w. \tag{13}$$

The determination of the temperature is realized through a similar procedure. Energy equation (11) at boundary conditions (12)–(13) is reduced to

$$T(r) = T_w + \left(\frac{dp/dz}{m+1}\right)^2 \int_r^R \frac{1}{k(T)r^m}\left(\int_0^r \frac{r^{m+2}}{\mu(T)}\,dr\right)dr \tag{14}$$

after the required substitutions for the velocity gradient are made and two integrations are performed.

In dimensionless form this solution becomes

$$\bar{T}(\rho) = 1 + M\int_\rho^1 \frac{1}{\bar{k}(\bar{T})^m}\left(\int_0^\rho \frac{\rho^{m+2}}{\bar{\mu}(\bar{T})}\,d\rho\right)d\rho \tag{15}$$

where $\bar{T}(\rho) = T(r)/T_w$, $\bar{k}(\bar{T}) = k(T)/k_0$,

$$M = \left(\frac{dp/dz}{m+1}\right)^2 \frac{R^4}{k_0\mu_0 T_w}.$$

Since $T(0) = T_m$, $\bar{T}(0) = \lambda$.

Thus,

$$\lambda = 1 + M\int_0^1 \frac{1}{\bar{k}(\bar{T})\rho^m}\left(\int_0^\rho \frac{\rho^{m+2}}{\bar{\mu}(\bar{T})}\,d\rho\right)d\rho. \tag{16}$$

Subtraction of equation (16) from equation (15) gives

$$\bar{T}(\rho) = \lambda - M\int_0^\rho \frac{1}{\bar{k}(\bar{T})\rho^m}\left(\int_0^\rho \frac{\rho^{m+2}}{\bar{\mu}(\bar{T})}\,d\rho\right)d\rho. \tag{17}$$

For constant phenomenological coefficients $\bar{k}(\bar{T}) = \bar{\mu}(\bar{T}) = 1$ the solution (17) gives

$$\bar{T}(\rho) = \lambda - (\lambda - 1)\rho^4 \tag{18}$$

where $\lambda = 1 + [M/4(m+3)]$.

We assume that both dynamic viscosity and thermal conductivity are taken in the form

$$\mu = \mu_0\left(\frac{T_0}{T}\right)^p, \quad k = k_0\left(\frac{T_0}{T}\right)^s \tag{19}$$

where μ_0 and k_0 are measured at a reference temperature of the fluid and the exponents p and s are constants.

In dimensionless form the phenomenological coefficients become

$$\bar{\mu}(\bar{T}) = \left(\frac{\lambda+1}{2\bar{T}}\right)^p, \quad \bar{k}(\bar{T}) = \left(\frac{\lambda+1}{2\bar{T}}\right)^s \tag{20}$$

Introduction of relation (20) into equations (9), (16) and (17) reduces these equations to

$$\bar{u}(\rho) = \int_\rho^1 \rho\bar{T}^p(\rho)\,d\rho \Big/ \int_0^1 \rho^{m+2}\bar{T}^p(\rho)\,d\rho \tag{21}$$

$$\bar{T}(\rho) = \lambda - M\left(\frac{2}{\lambda+1}\right)^{p+s}\varphi(\rho) \tag{22}$$

$$(\lambda - 1)(\lambda + 1)^{p+s} = 2^{p+s}M\varphi(1) \tag{23}$$

where

$$\varphi(\rho) = \int_0^\rho \frac{\bar{T}^s(\rho)}{\rho^m}\left(\int_0^\rho \rho^{m+2}\bar{T}^p(\rho)\,d\rho\right)d\rho. \tag{24}$$

Equation (23) gives the relationship between the flow parameter M and the temperature ratio λ.

3. ALGORITHM FOR SOLVING THE PROBLEM

Equation (22) is in a form amenable to solution by Picard's method of successive approximations. This method becomes very simple if we use the manipulation system FORMAL which permits one to integrate analytically. The following algorithm was successfully applied for solving the problem:

1. The calculation is started with the initial temperature distribution (18) corresponding to constant phenomenological coefficient

$$\bar{T}_0(\rho) = \lambda_0 - (\lambda_0 - 1)\rho^4 \tag{25}$$

where $\lambda_0 = 1 + [M/4(m+3)]$.

2. Substituting $\bar{T}_0(\rho)$ into equation (24)

$$\varphi_n(\rho) = \int_0^\rho \frac{\bar{T}_n^s(\rho)}{\rho^m}\left(\int_0^\rho \rho^{m+2}\bar{T}_n^p(\rho)\,d\rho\right)d\rho \tag{26}$$

one finds $\varphi_0(\rho)$ by direct analytical integration.

3. For selected value of the flow parameter M one can calculate λ_1 numerically from equation (23)

$$(\lambda_{n+1} - 1)(\lambda_{n+1} + 1)^3 = 8M\varphi_n(1). \tag{27}$$

4. $\bar{T}_1(\rho)$ is obtained by substituting $\varphi_0(\rho)$ and λ_1 into equation (22)

$$\bar{T}_{n+1}(\rho) = \lambda_{n+1} - M\left(\frac{2}{\lambda_{n+1}+1}\right)^{p+s}\varphi_n(\rho). \tag{28}$$

The result is re-substituted into equation (26) to yield $\bar{T}_2(\rho)$ and so on. The iteration order n was increased consequently up and stops when the accuracy required is achieved.

5. Using formulae (21) the velocity

$$\bar{u}_n(\rho) = \int_\rho^1 \rho\bar{T}_n^p(\rho)\,d\rho \Big/ \int_0^1 \rho^{m+2}\bar{T}_n^p(\rho)\,d\rho \tag{24}$$

is calculated analytically.

On the base of this algorithm the examples considered in [5] were recalculated.

4. ILLUSTRATIVE RESULTS

In [5] is studied the Poiseuille flow through a circular tube ($m = 1$) when the exponent of the phenomenological coefficient was: $p = 2$ and $s = 1$. Detailed data are presented for the temperature and velocity distributions for different values of the flow parameter Γ. It

Table 1. Temperature ratio λ as a function of the iteration order n

	Γ	
n	2	16
1	1.015 744 014	1.131 481 790
2	1.015 745 997	1.132 142 729
3	1.015 746 039	1.132 223 196
4	1.015 746 040	1.132 234 737
5	1.015 746 040	1.132 236 488
6		1.132 236 755

is easy to establish the following relation between the parameter Γ in [5] and M in the present paper: $\Gamma = 8M$.

The data in Table 1 show that six iterations are enough to assure for the temperature ratio to an accuracy of six figures after the decimal point.

Table 2 presents the convergence of the values of the temperature distribution $T_n(\rho)$ and the velocity profile $u(\rho)$ and can be considered as a test for the accuracy of the data given in [5], p. 1126.

The advantage of the solution with symbol manipulation system FORMAL is that the result is obtained directly in a polynomial expansion

$$\bar{T}(\rho) = \sum(-1)^n a_n \rho^{4n} \quad (25)$$

which easily permits the calculation of data for any ρ.

The temperature ratios presented in Table 1 are calculated using the system FORMAL realised on computer RC 4000 for 4 ($\Gamma = 2$) and 6 ($\Gamma = 16$) seconds.

The examples considered above show that the analytical–numerical methods realized through formula manipulation techniques can successfully be used and one can expect to find their recent application in heat- and mass-transfer problems.

REFERENCES

1. J. E. Sammet, Revised annotated discriptor based bibliography on the use of computers for non-numerical mathematics, *Symbol Manipulation Languages and Techniques*, edited by D. Bobrov, pp. 358–484. North Holland, Amsterdam (1971).
2. D. Barton and J. P. Fitch, Applications of algebraic manipulative programs in physics, *Rep. Prog. Phys.* **35**, 235–314 (1972).
3. Papers from the First Symposium on Symbolic and Algebraic Manipulation (SYM SAM/1), *Communs Ass. Comput. Mech.* **9**(8) (1966).
4. Papers from the Second Symposium on Symbolic and Algebraic Manipulation (SYM SAM/2), *Communs Ass. Comput. Mech.* **14**(8) (1971).
5. H. W. Butler and R. Rackley, Application of a variational formulation to nonequilibrium fluid flow, *Int. J. Heat Mass Transfer* **10**, 1255–1266 (1967).

Table 2. Dimensionless temperatures $\bar{T}_n(\rho)$ and velocity profiles as a function of the radius and the flow parameter ($m = 1, p = 2, s = 1$)

Γ	ρ	$\bar{T}_1(\rho)$	$\bar{T}_2(\rho)$	$\bar{T}_3(\rho)$	$\bar{T}_4(\rho)$	$\bar{T}_5(\rho)$	$\bar{T}_6(\rho)$	$\bar{U}_6(\rho)$
2	0.0	1.015 744	1.015 746	1.015 746	1.015 746	1.015 746		2.0108
	0.1	1.015 742	1.015 744	1.015 744	1.015 744	1.014 744		1.9904
	0.2	1.015 718	1.015 720	1.015 720	1.015 720	1.015 720		1.9295
	0.3	1.015 615	1.015 616	1.015 617	1.015 617	1.015 617		1.8279
	0.4	1.015 335	1.015 337	1.015 337	1.015 337	1.015 337		1.6858
	0.5	1.014 746	1.014 747	1.014 747	1.014 747	1.014 747		1.5031
	0.6	1.013 676	1.013 677	1.013 677	1.013 677	1.013 677		1.2803
	0.7	1.011 919	1.011 920	1.011 920	1.011 920	1.011 920		1.0177
	0.8	1.009 236	1.009 237	1.009 237	1.009 237	1.009 237		0.7160
	0.9	1.005 359	1.005 360	1.005 360	1.005 360	1.005 360		0.3762
	1.0	1.000 000	1.000 000	1.000 000	1.000 000	1.000 000		0.0000
16	0.0	1.131 482	1.132 143	1.132 223	1.132 235	1.132 236	1.132 237	2.0856
	0.1	1.131 467	1.132 128	1.132 208	1.132 220	1.132 222	1.132 222	2.0630
	0.2	1.131 247	1.131 904	1.131 984	1.131 995	1.131 997	1.131 997	1.9951
	0.3	1.130 292	1.130 933	1.131 012	1.131 023	1.131 025	1.131 025	1.8820
	0.4	1.127 728	1.128 330	1.128 404	1.128 415	1.128 416	1.128 417	1.7242
	0.5	1.122 356	1.122 877	1.122 944	1.122 954	1.122 955	1.122 955	1.5227
	0.6	1.112 699	1.113 095	1.113 149	1.113 157	1.113 159	1.113 159	1.2797
	0.7	1.097 110	1.097 348	1.097 386	1.097 392	1.097 393	1.097 393	0.9990
	0.8	1.073 943	1.074 033	1.074 055	1.074 058	1.074 059	1.074 059	0.6865
	0.9	1.041 822	1.041 832	1.041 841	1.041 842	1.041 843	1.041 843	0.3501
	1.0	0.999 996	0.999 996	0.999 996	0.999 996	0.999 996	0.999 996	0.0000

ETUDE DE L'ECOULEMENT D'UN FLUIDE HORS D'EQUILIBRE A L'AIDE D'UNE METHODE ANALYTIQUE

Résumé—On résout le problème d'un écoulement non isotherme entre plaques parallèles et à l'intérieur d'un tube circulaire dans le cas de la conductivité thermique et de la viscosité fonctions de la température. Les solutions sont obtenues par une méthode analytique.

DIE ANWENDUNG MATHEMATISCHER UMFORMUNGEN AUF NICHTGLEICHGEWICHTS-STRÖMUNGEN

Zusammenfassung—Die Probleme der nichtisothermen Strömung zwischen parallelen Platten und in Rohren werden für den Fall temperaturabhängiger Werte der Wärmeleitfähigkeit und der Viskosität gelöst. Die Lösungen werden mit Hilfe mathematischer Umformungen gewonnen.

ИСПОЛЬЗОВАНИЕ МЕТОДА ПРЕОБРАЗОВАНИЯ ФОРМУЛ В СЛУЧАЕ НЕРАВНОВЕСНОГО ТЕЧЕНИЯ ЖИДКОСТИ

Аннотация — Дано решение задач неизотермического течения жидкости между параллельными пластинами и в круглой трубе для случая зависимости теплопроводности и вязкости от температуры. Решения получены методом преобразования формул.